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ON DISTANCE SETS OF LARGE SETS OF INTEGER POINTS*

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ABSTRACT

Distance sets of large sets of integer points are studied in dimensions at least 5. To any $\varepsilon > 0$ a positive integer Q_ε is constructed with the following property; If A is any set of integer points of upper density at least ε, then all large multiples of Q_ε^2 occur as squares of distances between the points of the set A.

1. Introduction.

A result of Furstenberg, Katznelson and Weiss [FKW] states that if A is a measurable subset of $\mathbb{R}^2$ of positive upper density, then its distance set: $d(A) = \{|x-y| : x \in A,\ y \in A\}$ contains all large numbers.

Our aim is to prove a similar result for subsets of $\mathbb{Z}^n$ $(n > 4)$ of positive density ε, namely that: $d^2(A) = \{|m-l|^2 : m \in A,\ l \in A\}$ contains all large multiples of a fixed number Q_ε^2, which depends only on the density ε and the dimension n.

Note that one cannot take $Q_\varepsilon = 1$ as the set A may fall into a fixed congruence class of some integer q, and if $q \leq \varepsilon^{-1/n}$ then such a set A would have density $q^{-n} \geq \varepsilon$, and all elements of $d^2(A)$ would be divisible by q^2. Moreover, this implies that Q'_ε divides Q_ε, where Q'_ε is the least common multiple of all $q \leq \varepsilon^{-1/n}$. In particular, by the prime number theorem $Q_\varepsilon \geq Q'_\varepsilon \geq \exp(c\,\varepsilon^{-1/n})$

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with some $c > 0$. The number Q_ε we construct will be similar and will satisfy the upper bound: $Q_\varepsilon \leq \exp(C_n\, \varepsilon^{-6/n-4})$, where C_n is a constant depending only on the dimension n.

Such results are impossible in dimensions $n \leq 3$. Indeed, even if one takes $A = \mathbb{Z}^n$, the equation: $d = |m-l|^2$ has no solution if $d = 4^a(8k+1)$ by Gauss' characterization, however, every number has multiples of this form. We prove our result in dimensions $n \geq 5$ leaving the case $n = 4$ open.

A corollary is that the gaps between consecutive distances $d < d'$, $d,\ d' \in d(A)$ satisfy: $d' - d \leq C_\varepsilon d^{-1/2}$ where ε denotes the upper density of the set A. Distance sets of discrete subsets of $\mathbb{R}^n$ have been studied before, in [IŁ] it was shown that the gaps between consecutive distances from discrete subsets A of $\mathbb{R}^2$ tend to 0, if A has a point in every square of size $\sqrt{5}$. In fact, this was proved more generally when the distances are associated to convex sets, see also [K] for similar higher dimensional results. Our proof may be generalized when the distances are defined by certain positive homogeneous polynomials, however we do not pursue such generalizations here.

2. Main results.

We say that a set $A \subseteq \mathbb{Z}^n$ has upper density at least ε, and write $\delta(A) \geq \varepsilon$, if there exists a sequence of cubes B_{R_j} of sizes $R_j \to \infty$, not necessarily centered at the origin, such that

$$|A \cap B_{R_j}| \geq \varepsilon\, R_j^n \quad \text{for all } j, \tag{2.1}$$

where $|A|$ denotes the number of elements of the set A. As usual $\mathbb{Z}^n$ denotes the integer lattice and $\mathbb{N}$ stands for the natural numbers.

THEOREM 1: *Let $n \geq 5$, $\varepsilon > 0$ and let $A \subseteq \mathbb{Z}^n$ such that $\delta(A) \geq \varepsilon$.*

Then there exists $Q_\varepsilon \in \mathbb{N}$ depending only on ε, and $\Lambda_A \in \mathbb{N}$ depending on the set A, such that

$$\lambda Q_\varepsilon^2 \in d^2(A) = \{|m-l|^2 : m \in A,\ l \in A\} \tag{2.2}$$

for every $\lambda \geq \Lambda_A$.

In fact, a more quantitative version will be proved

THEOREM 2: *Let $n \geq 5$, $\varepsilon > 0$. Then there exist a pair $J_\varepsilon, Q_\varepsilon \in \mathbb{N}$ depending only on ε, such that the following holds.*

If $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{J_\varepsilon}$ *is any sequence of natural numbers with* $\lambda_{j+1} \geq 10\lambda_j$ *and if* $A \subseteq \mathbb{Z}^n \cap B_R$ *such that* $|A| \geq \varepsilon R^n$ *and* $R \geq 10\lambda_{J_\varepsilon}^{1/2}$, *then*

$$\exists\, j \leq J_\varepsilon \quad \text{such that } \lambda_j Q_\varepsilon^2 \in d^2(A) \tag{2.3}$$

It is clear that Theorem 2 implies Theorem 1. Indeed, let $\varepsilon > 0$, J_ε, Q_ε as in Theorem 2. If Theorem 1 does not hold for Q_ε, then there is a set $A \subseteq \mathbb{Z}^n$ with upper density $\delta(A) \geq \varepsilon$ and infinite sequence λ_j such that $10\lambda_j < \lambda_{j+1}$ and $\lambda_j Q_\varepsilon^2 \notin d^2(A)$ for all j. Choosing a cube B_R with size $R > 10\lambda_{J_\varepsilon}^{1/2}$ such that $|A \cap B_R| \geq \varepsilon R^n$ contradicts (2.3).

It will be convenient to introduce the following terminology; a triple (ε, Q, J) is called **regular** if the conclusion of Theorem 2 holds for that triple. It is clear that the triple (1,1,1) is regular as every positive integer is the sum of 5 squares, and also the regularity of (ε, Q, J) implies that of (ε', Q, J) for $\varepsilon \leq \varepsilon'$.

Thus it is enough to show that for each $\varepsilon_k = (9/10)^k$ there exists a pair of natural numbers Q_k, J_k such that $(\varepsilon_k, Q_k, J_k)$ is regular. This will be shown by induction on k, constructing $Q_k = Q_{k-1}q_k$ and J_k recursively to ε_k. The point is that induction will enable one to assume that A is well-distributed in the congruence classes of a fixed modulus q_k, to be chosen later.

Indeed for $s \in \mathbb{Z}^n$, let $A_{q_k,s} = \{m \in \mathbb{Z}^n : q_k m + s \in A\}$. If there is an s such that the density $\delta(A_{q_k,s}) \geq \varepsilon_{k-1}$, then by induction it follows that $\lambda Q_{k-1}^2 \in d^2(A_{q_k,s})$ for all large λ, and hence $\lambda Q_k^2 \in d^2(A)$. Thus one can assume that $\delta(A) \geq \varepsilon_k$, but $\delta(A_{q_k,s}) \leq \varepsilon_{k-1} = (10/9)\varepsilon_k$ for each $s \in \mathbb{Z}^n$.

Our proof was motivated by the short Fourier analytic proof of the Fürstenberg-Katznelson-Weiss theorem given in [B]. The starting point is to express the number of pairs $m \in A,\ l \in A$ such that $|m-l|^2 = \lambda$ in the form

$$N(A,\lambda) = \sum_{m,l} 1_A(m) 1_A(l)\, \sigma_\lambda(m-l) = \langle 1_A, 1_A * \sigma_\lambda \rangle$$

where 1_A denotes the indicator function of the set A and σ_λ stands for that of the of the set of integer points on the sphere of radius $\lambda^{1/2}$. Thus by Plancherel

$$N(A,\lambda) = \langle \hat{1}_A, \hat{1}_A\, \hat{\sigma}_\lambda \rangle = \int_{\Pi^n} |\hat{1}_A(\xi)|^2 \hat{\sigma}_\lambda(\xi)\, d\xi, \tag{2.4}$$

where

$$\hat{\sigma}_\lambda(\xi) = \sum_{|m|^2=\lambda} e^{2\pi i m\cdot\xi} \tag{2.5}$$

is the Fourier transform of σ_λ, and Π^n denotes the n-dimensional torus.

Note that $|\hat\sigma_\lambda(\xi)| \le \hat\sigma_\lambda(0) \lesssim \lambda^{n/2-1}$ for $n \ge 5$, using the well-known fact in number theory, that $|\{m \in \mathbb{Z}^n : |m|^2 = \lambda\}| \lesssim \lambda^{n/2-1}$. Thus if $A \subseteq B_R$ then by (2.4):

$$N(A,\lambda) \lesssim |A|\lambda^{n/2-1} \le R^n \lambda^{n/2-1}. \tag{2.6}$$

Here $A \lesssim B$ means that $A \le c_n B$ with a constant $c_n > 0$ depending only on the dimension n, and whose exact value may change from place to place.

The behavior of the exponential sum $\hat\sigma_\lambda(\xi)$ is described in [MSW] and summarized in the asymptotic formula (3.3). We will use the fact that it is concentrated near rational points of small denominator. More precisely, given $\varepsilon > 0$, there is a $Q_\varepsilon \in \mathbb{N}$ and a $\lambda_\varepsilon > 0$ depending only on ε, such that for $\lambda \ge \lambda_\varepsilon$

$$|\hat\sigma_\lambda(\xi)| \lesssim \varepsilon^3 \lambda^{\frac{n}{2}-1} \quad \text{if } |\xi - l/Q_\varepsilon| \gtrsim \varepsilon^{-6/(n-1)}\,\lambda^{-1/2}$$

for every rational point l/Q_ε, $l \in \mathbb{Z}^n$. This implies

$$\int_{\Pi^n} |\hat 1_A(\xi)^2\,\hat\sigma_\lambda(\xi)\,\Big(1 - \sum_{l\in\mathbb{Z}^n} \hat\psi^1_\lambda(\xi - l/Q_\varepsilon)\Big)|\,d\xi \lesssim \varepsilon^3 R^n \lambda^{n/2-1} \tag{2.7}$$

where $0 \le \hat\psi^1_\lambda(\xi) \le 1$ is smooth cut-off function, such that $\hat\psi^1_\lambda(0) = 1$ and is supported on the ball $|\xi| \lesssim \varepsilon^{-6/(n-1)}\,\lambda^{-1/2}$. This will be proved in Section 3.

In Section 4, we prove our key estimate, namely that if $A \subseteq B_R$, $|A| \ge \varepsilon R^n$ and if A is uniformly distributed in the congruence classes of a certain modulus q_ε, then

$$\int_{\Pi^n} |\hat 1_A(\xi)|^2 \hat\sigma_\lambda(\xi) \sum_{l\in\mathbb{Z}^n} \hat\psi^2_\lambda(\xi - l/Q_\varepsilon)\,d\xi \gtrsim \varepsilon^3 R^n \lambda^{n/2-1}, \tag{2.8}$$

where ψ^2_λ is a smooth function whose Fourier transform $\hat\psi^2_\lambda(\xi)$ is supported on $|\xi| \lesssim \lambda^{-1/2}$.

Now assume that, in the settings of Theorem 2, $N(A,\lambda) = 0$ for each $\lambda = \lambda_j Q_\varepsilon^2$ $(J_\varepsilon/2 \le j \le J_\varepsilon)$. Using the decomposition

$$1 = \Big(1 - \sum_{l\in\mathbb{Z}^n} \hat\psi^1_\lambda(\xi - l/Q_\varepsilon)\Big) + \sum_{l\in\mathbb{Z}^n} \hat\psi^2_\lambda(\xi - l/Q_\varepsilon) + \sum_{l\in\mathbb{Z}^n} \hat\phi_\lambda(\xi - l/Q_\varepsilon)$$

in (2.4), it follows from (2.7) and (2.8) and from the uniform bound: $|\hat\sigma_\lambda(\xi)| \lesssim \lambda^{n/2-1}$, that

$$\int_{\Pi^n} |\hat 1_A(\xi)|^2 \Big|\sum_{l\in\mathbb{Z}^n} \hat\phi_\lambda(\xi - l/Q_\varepsilon)\Big|\,d\xi \gtrsim \varepsilon^3 R^n, \tag{2.9}$$

where $\hat{\phi}_\lambda(\xi) = \hat{\psi}^1_\lambda(\xi) - \hat{\psi}^2_\lambda(\xi)$ is a smooth function essentially supported on the annulus:

$$\lambda^{-1/2} \leq |\xi| \leq \varepsilon^{-6/(n-1)}\,\lambda^{-1/2}.$$

Thus the supports of the integrands in (2.9) for well-separated values of $\lambda = \lambda_j Q_\varepsilon^2$ are (essentially) disjoint, and hence the sum of the left side (2.9) over such j's would be bounded by $\int_{\Pi^n} |\hat{1}_A(\xi)|^2\, d\xi = |A| \leq R^n$ which contradicts (2.9) if J_ε is chosen large enough.

3. Upper bounds.

In this section we prove the upper bound (2.7), which is an easy corollary of the following asymptotic formula, proved in [MSW] (see Proposition 4.1):

$$\text{(3.1)}\quad \hat{\sigma}_\lambda(\xi) = \omega_n \lambda^{n/2-1} \sum_{r=1}^{\infty} m_{r,\lambda}(\xi) + E(\xi,\lambda) \quad \text{where} \quad \sup_\xi |E(\xi,\lambda)| \lesssim \lambda^{n/4};$$

and the main terms are of the form

$$\text{(3.2)}\qquad m_{r,\lambda}(\xi) = \sum_{k\in\mathbb{Z}^n} S(k,r,\lambda)\,\phi(r\xi - k)\,d\tilde{\sigma}(\lambda^{1/2}(\xi - k/r)).$$

Moreover, by the standard estimate for Gauss sums

$$\text{(3.3)}\qquad |S(k,r,\lambda)| = |q^{-n} \sum_{(a,r)=1} \sum_{s\in\mathbb{Z}_r^n} e^{2\pi i \frac{a(|s|^2-\lambda)+s\cdot l}{r}}| \leq r^{-n/2+1}.$$

The cut-off function $\phi(\xi)$ is supported in a small neighborhood of the origin, and the Fourier transform of the surface area measure of the unit sphere satisfies

$$\text{(3.4)}\qquad |d\tilde{\sigma}(\xi)| \lesssim (1+|\xi|)^{-(n-1)/(2)}.$$

Proposition 1: Let $n \geq 5$, $\varepsilon > 0$ and $c > 0$ be given. Then there is a constant $C_n > 0$ depending only on c and the dimension n, such that the following holds.

If $Q \in \mathbb{N}$ is such that r divides Q for all $r \leq C_n\,\varepsilon^{-6(n-4)}$, moreover if $\lambda \geq C_n\,\varepsilon^{-12/(n-4)}$ and $\xi \in \Pi^n$ satisfies

$$\text{(3.5)}\qquad |\xi - l/Q| \geq C_n\,\varepsilon^{-6/(n-1)}\lambda^{-1/2},$$

for all $l \in \mathbb{Z}^n$, then one has

$$\text{(3.6)}\qquad |\hat{\sigma}_\lambda(\xi)| \leq c\,\varepsilon^3 \lambda^{n/2-1}.$$

Proof. By equation (3.1), one has for $n \geq 5$

$$|E(\xi,\lambda)| \leq c_1 \lambda^{n/4} \leq (c/3)\,\varepsilon^3 \lambda^{n/2-1} \tag{3.7}$$

if $\lambda \geq C_n\,\varepsilon^{-\frac{12}{n-4}}$ and C_n is large enough with respect to c_1, c and n. There is at most one non-zero term in the expression for $m_{r,\lambda}(\xi)$, hence by (3.3)

$$\sum_{r\geq K} |m_{r,\lambda}(\xi)| \leq c_2 K^{-n/2+2} \leq \,(c/3)\,\varepsilon^3 \tag{3.8}$$

if one chooses $K \geq C_n\,\varepsilon^{-6/(n-4)}$.

If Q is such that r divides Q for all $r \leq K$, then every rational point k/r can be written in the form l/Q, then (3.4) and (3.5) implies

$$\sum_{r\leq K} |m_{r,\lambda}(\xi)| \leq c_3 \max_{k,\,r\leq K} |\lambda^{1/2}(\xi - k/r)|^{-(n-1)/2} \leq \,(c/3)\,\varepsilon^3. \tag{3.9}$$

The Proposition follows by adding (3.7)–(3.9) ∎

In what follows, C_n will denote a large enough constant of our choice, which guarantees the validity of certain inequality, and whose exact value may change from place to place.

The above estimate shows that $\hat{\sigma}_\lambda(\xi)$ is uniformly small on the complement of the neighborhoods $U_l = \{\xi :\ |\xi - l/Q_\varepsilon| \geq C_n\,\varepsilon^{-6/(n-1)}\lambda^{-1/2}\}$. Thus it can be used to bound the contribution of this set to the integral $N(A,\lambda)$ given in (2.4). To be more precise, let $A \subseteq B_R$ such that $|A| \geq \varepsilon R^n$, and let $\psi > 0$ be a smooth function satisfying

$$1 = \hat{\psi}(0) \geq \hat{\psi}(\xi) > 0 \quad \text{for all } \xi \quad \text{and} \quad \operatorname{supp}\hat{\psi} \subseteq [-1/2,1/2]^n\,. \tag{3.10}$$

For $Q \in \mathbb{N}$ and $L > 0$ define the following expressions

$$N_1(A,\lambda,Q,L) = \int_{\Pi^n} |\hat{1}_A(\xi)|^2\,\hat{\sigma}_\lambda(\xi)\left(1 - \sum_{l\in\mathbb{Z}^n} \hat{\psi}(L(\xi - l/Q))\right) d\xi \tag{3.11}$$

$$N_2(A,\lambda,Q,L) = \int_{\Pi^n} |\hat{1}_A(\xi)|^2\,\hat{\sigma}_\lambda(\xi) \sum_{l\in\mathbb{Z}^n} \hat{\psi}(L(\xi - l/Q))\,d\xi.$$

Lemma 1: Let $\varepsilon > 0$ and $c > 0$ be given. Then there is a constant $C_n > 0$, such that if $Q \in \mathbb{N}$ is a multiple of all $r \leq C_n\,\varepsilon^{-6/(n-4)}$, moreover, if $L \geq C_n\,Q$ and $\lambda \geq C_n\,\varepsilon^{-12/(n-1)-3}L^2$ then

$$|N_1(A,\lambda,Q,L)| \leq c\,\varepsilon^3\lambda^{n/2-1}R^n. \tag{3.12}$$

Proof. Since $\int_{\Pi^n} |\hat{1}_A(\xi)|^2 \, d\xi = |A| \le R^n$ it is enough to show that

$$\sup_{\xi \in \Pi^n} |\hat{\sigma}_\lambda(\xi)| \left| 1 - \sum_{l \in \mathbb{Z}_Q^n} \hat{\psi}(L(\xi - l/Q) \right| \le c\, \varepsilon^3 \lambda^{n/2-1} \tag{3.13}$$

Note that the supports of the functions $\hat{\psi}(L(\xi - l/Q))$ are disjoint for different values of l, thus if there is an l_0 such that: $|\xi - l_0/Q| \le c_1\, \varepsilon^{\frac{3}{2}} L^{-1}$, where c_1 is small enough with respect to c, then

$$\left| 1 - \sum_{l \in \mathbb{Z}^n} \hat{\psi}(L(\xi - l/Q)) \right| = |1 - \hat{\psi}(L(\xi - l_0/Q))| \le c\, \varepsilon^3$$

using the fact that $|1 - \hat{\psi}(\eta)| \lesssim |\eta|^2$ and (3.13) follows. In the opposite case:

$$|\xi - l/Q| > c_1\, \varepsilon^{3/2} L^{-1} \ge C_n \varepsilon^{-\frac{6}{n-1}} \lambda^{-1/2}.$$

for all $l \in \mathbb{Z}^n$, by the assumptions on L and λ, and (3.13) follows from (3.6). ∎

4. Lower bounds.

In this section we prove the lower bound (2.8). We start by proving an analogous estimate in the settings of the group of congruence classes of the modulus Q: $\mathbb{Z}_Q^n = (\mathbb{Z}/Q\mathbb{Z})^n$. Let $\varepsilon > 0$ and let $q,\ Q \in \mathbb{N}$ such that q divides Q. Let $f : \mathbb{Z}_Q^n \to [0,1]$ be a function satisfying the following two conditions

$$\sum_{m \in \mathbb{Z}_Q^n} f(m) \ge \frac{4\varepsilon}{5}\, Q^n \tag{4.1}$$

and for each $s \in \mathbb{Z}_Q^n$

$$\sum_{m \in \mathbb{Z}_Q^n} f(qm + s) \le \frac{10\varepsilon}{9}\, Q^n \tag{4.2}$$

Note that if f is the characteristic function of a set $A \subseteq \mathbb{Z}_Q^n$ then by equation (4.1) the density of A is at least $4\varepsilon/5$, while by (4.2) the density of A is at most $10\varepsilon/9$ in any of the congruence classes of the modulus q. We say that the set A is well-distributed in these congruence classes. For $\lambda \in \mathbb{Z}$ we consider the following quantity:

$$N = N(f, Q, \lambda) = \sum_{m, l \in \mathbb{Z}_Q^n} f(m) f(m-l)\, \omega_{\lambda,Q}(l), \tag{4.3}$$

where the function: $\omega_{\lambda,Q} : \mathbb{Z}_Q^n \to \mathbb{N}$ is defined by

$$\omega_{\lambda,Q}(l) = |\{k \in \mathbb{Z}^n : |k|^2 = \lambda,\ k \equiv l \pmod{Q}\} \tag{4.4}$$

that is the number of lattice points of length $\lambda^{1/2}$ which are congruent to l modulo Q. We will make use of the Fourier transform on $\mathbb{Z}_Q^n$:

$$\hat{f}(s) = \sum_{m \in \mathbb{Z}_Q^n} e^{-2\pi i \frac{m\cdot s}{Q}} f(m)$$

and note that

$$\hat{\omega}_{\lambda,Q}\,(s) = \sum_{k \in \mathbb{Z}^n,\ |k|^2=\lambda} e^{-2\pi i \frac{k\cdot s}{Q}} = \hat{\sigma}_\lambda(s/Q). \tag{4.5}$$

Lemma 2: Let $0 < \varepsilon < 1$ and let q, Q, λ be positive integers such that k divides q for all $k \leq C_n\,\varepsilon^{-\frac{6}{n-4}}$, q divides Q, and $\lambda > C_n\,Q^2\,\varepsilon^{-\frac{12}{n-1}}$.

If $f\colon \mathbb{Z}_Q^n \to [0,1]$ is a function satisfying (4.1) and (4.2) then for C_n large enough, one has

$$N(f, Q, \lambda) \geq c\,\lambda^{n/2-1} Q^n \varepsilon^2, \tag{4.6}$$

where $c > 0$ is a constant depending only on the dimension n.

Proof. Using the Fourier transform on $\mathbb{Z}_Q^n$, similarly as in (2.4), one has

$$N = N(f, Q, \lambda) = \frac{1}{Q^n} \sum_{s \in \mathbb{Z}_Q^n} |\hat{f}(s)|^2\,\hat{\sigma}_\lambda(s/Q).$$

Write $Q = Q_1 q$ and decompose the summation into two terms according to whether Q_1 divides s;

$$N = \frac{1}{Q^n} \sum_{s_1 \in \mathbb{Z}_q^n} |\hat{f}(Q_1 s_1)|^2\,\hat{\sigma}_\lambda(s_1/q) + \frac{1}{Q^n} \sum_{Q_1 \nmid s} |\hat{f}(s)|^2\,\hat{\sigma}_\lambda(s/Q) = M + E.$$

Here the main term M is obtained by writing $s = Q_1 s_1$ where s_1 is running through $\mathbb{Z}_q^n$. Let $f_q\colon\ \mathbb{Z}_q^n \to [0,1]$ be defined by:

$$f_q(m) = Q_1^{-n} \sum_{k \in \mathbb{Z}_{Q_1}^n} f(m + qk),$$

that is the average of f over the congruence class of m with respect to the modulus q. Then $\hat{f}_q(s) = Q_1^{-n}\hat{f}(Q_1 s)$, thus by equation (4.5) and Plancherel

$$M = \frac{Q_1^{2n}}{Q^n} \sum_{s_1 \in \mathbb{Z}_q^n} |\hat{f}_q(s_1)|^2\,\hat{\sigma}_\lambda(s_1/q) \ = \ Q_1^n \sum_{m,l \in \mathbb{Z}_q^n} f_q(m) f_q(l)\,\omega_{\lambda,q}(m - l). \tag{4.7}$$

Note that (4.1) and (4.2) is equivalent to

$$\sum_{m\in\mathbb{Z}_q^n} f_q(m) \geq \frac{4\varepsilon}{5}\, q^n \quad \text{and} \quad f_q(m) \leq \frac{10\varepsilon}{9}\;, \quad \text{for all } m \in \mathbb{Z}_q^n. \tag{4.8}$$

If $G = \{m \in \mathbb{Z}_q^n : f_q(m) \geq \varepsilon/10\,\}$ then the sum of $f_q(m)$ for $m \notin G$ is at most $q^n\varepsilon/10$ thus by (4.8)

$$|G| \geq \frac{9}{10\varepsilon} \sum_{m\in G} f_q(m) \geq \frac{63}{100}\, q^n. \tag{4.9}$$

Hence for every $l \in \mathbb{Z}_q^n$,

$$\sum_{m\in\mathbb{Z}_q^m} f_q(m)\, f_q(m-l) \geq \frac{\varepsilon^2}{100}|G \cap (G+l)| > \frac{\varepsilon^2}{500}\, q^n.$$

Substituting back to (4.7) one has

$$M \geq \frac{\varepsilon^2 Q^n}{500} \sum_{l\in\mathbb{Z}_q^n} \omega_{\lambda,q}(l) = \frac{\varepsilon^2 Q^n}{500}\; \hat{\sigma}_\lambda(0) \geq c_n\, Q^n \varepsilon^2 \lambda^{n/2-1}, \tag{4.10}$$

where the constant $c_n > 0$ depends only on the dimension n.

Now let C_n be chosen as in Proposition 1. with respect to $c = c_n/2$. If Q_1 does not divide s, then for every $l \in \mathbb{Z}^n$

$$\Big|\frac{s}{Q} - \frac{l}{q}\Big| \geq \frac{1}{Q} \geq \varepsilon^{-\frac{6}{n-1}}\, \lambda^{-\frac{1}{2}}$$

The conditions of Lemma 1. are satisfied, thus

$$|\hat{\sigma}_\lambda(s/Q)| \leq \frac{c_n}{2}\, \varepsilon^3\, \lambda^{\frac{n}{2}-1},$$

hence by Plancherel $|E| \leq \frac{c_n}{2}\, \varepsilon^3 Q^n \lambda^{n/2-1}$, and the lemma follows with $c = c_n/2$. ∎

Next, our aim is to reduce estimate (2.8) to that of (4.6). First, one has

Proposition 2: Let $Q, \lambda \in \mathbb{N}$ and let $L \geq \lambda^{1/2}$. Then one has

$$N_2(A,\lambda,Q,L) \geq c_n\, Q^n\, L^{-n} \sum_{m\in\mathbb{Z}^n} \sum_{\substack{cl\in\mathbb{Z}^n\\ |l|\leq\sqrt{n}L}} 1_A(m)\, 1_A(m-l)\, \omega_{\lambda,Q}(l). \tag{4.11}$$

Proof. Define the distribution δ_Q by

$$\langle \delta_Q, \phi\rangle = \sum_{m\in\mathbb{Z}^n} Q^n\, \phi(Qm),$$

then by Poisson summation

$$\langle\hat{\delta}_Q, \phi\rangle = \langle\delta_Q, \hat{\phi}\rangle = \sum_{l\in\mathbb{Z}^n} \phi(l/Q). \tag{4.12}$$

Thus by Plancherel

$$N_2(A,\lambda,Q,L) = \langle\hat{1}_A, \hat{1}_A\hat{\sigma}_\lambda(\hat{\psi}_L * \hat{\delta}_Q)\rangle = \langle 1_A, 1_A * \sigma_\lambda * (\psi_L\delta_Q)\rangle,$$

where $\psi_L(x) = L^{-n}\psi(x/L)$. If $l \in \mathbb{Z}^n$ such that $|l| \leq \sqrt{n}L$ then

$$\begin{aligned}\sigma_\lambda * (\psi_L\delta_Q)(l) &= \sum_{k\in\mathbb{Z}^n} \sigma_\lambda(k)\,\psi_L(l-k)\,\delta_Q(l-k)\\ &\geq c_n\, Q^n L^{-n} \sum_{k:\ Q|l-k} \sigma_\lambda(k) = c_n Q^n L^{-n}\omega_{\lambda,Q}(l).\end{aligned}$$

Indeed, if $\sigma_\lambda(k) \neq 0$, then $|k| = \lambda^{1/2} \leq L$; hence $|l-k| \leq (\sqrt{n}+1)L$ and $\psi_L(l-k) \geq c_n L^{-n}$. This proves the Proposition. ∎

Assume that $R > 10\lambda^{1/2}$, choose L such that $\lambda^{1/2} \leq L \leq 2\lambda^{1/2}$ and R/L is an integer. Divide the box B_R into R^n/L^n boxes $B_L(j)$ of equal size L, and let $\mathcal{F}$ denote the set of boxes in which the density of A remains large:

$$\mathcal{F} = \{j : |A \cap B_L(j)| \geq (4\varepsilon)/5\, L^n\}. \tag{4.13}$$

It is easy to see that $|\mathcal{F}| \geq \frac{\varepsilon}{10}\frac{R^n}{L^n}$. If f_j denotes the characteristic function of the set $A \cap B_L(j)$, then by (4.12)

$$N_2(A,\lambda,Q,L) \geq c_n\, Q^n L^{-n} \sum_{j\in\mathcal{F}} \sum_{m,l\in\mathbb{Z}^n} f_j(m)\, f_j(m-l)\,\omega_{\lambda,Q}(l), \tag{4.14}$$

since the diameter of each box $B_L(j)$ is at most $\sqrt{n}\,L$.

The function $\omega_{\lambda,Q}$ is constant on the congruence classes mod Q, hence the inner sum in (4.15) can be written in the form

$$\frac{L^{2n}}{Q^{2n}} \sum_{m,l\in\mathbb{Z}_Q^n} f_{j,Q}(m)\, f_{j,Q}(m-l)\,\omega_{\lambda,Q}(l) \quad \text{where}$$

$$f_{j,Q}(m) = \sum_{k\in\mathbb{Z}^n} f_j(Qk+m) = \frac{Q^n}{L^n}\,|\{m' \in A \cap B_L(j) : m' \equiv m \pmod{Q}\}|$$

If one assumes that $A \cap B_L(j)$ is well-distributed in the congruence classes mod q, that is if for every $s \in \mathbb{Z}^n$ and $j \in \mathcal{F}$

$$\sum_{m\in\mathbb{Z}_Q^n} f_{j,Q}\,(qm+s) = |\{m \in A \cap B_L(j) :\ m \equiv s \pmod{q}\}| \leq \frac{10\varepsilon}{9}\frac{L^n}{q^n}, \tag{4.15}$$

then the functions $f_{j,Q}$ satisfy (4.1) and (4.2)

$$\sum_{m\in\mathbb{Z}^n_Q} f_{j,Q}\,(m) = \frac{Q^n}{L^n}\,|A\cap B_L(j)| \geq \frac{4\varepsilon}{5}\,Q^n$$

$$\sum_{m\in\mathbb{Z}^n_Q} f_{j,Q}\,(qm+s) = \frac{Q^n q^n}{L^n}\sum_{k\in\mathbb{Z}^n} f_j(qk+s) \leq \frac{10\varepsilon}{9}\,Q^n.$$

If the parameters q, Q, λ satisfy the conditions of Lemma 2, then from (4.6) and (4.15) one obtains the lower bound

$$N_2(A,\lambda,Q,L) \,\geq\, c_n\,Q^{-n}L^n\,|\mathcal{F}|\,\varepsilon^2 Q^n\lambda^{\frac{n}{2}-1} \geq\, c_n\,\varepsilon^3 R^n\lambda^{\frac{n}{2}-1}. \tag{4.16}$$

5. Proof of Theorem 2.

We are in a position to apply an induction argument to prove our main result. Let $\varepsilon_0 = Q_0 = J_0 = 1$ and for $k = 1, 2, \ldots$, define

$$\varepsilon_k = (9/10)^k\ , \quad q_k = \left[C_n\,\varepsilon_k^{-\frac{6}{n-4}}\right]!! \quad \text{and} \quad Q_k = q_k Q_{k-1}. \tag{5.1}$$

Moreover, we define J_k be the smallest integer satisfying

$$J_k \geq 2J_{k-1} + C_n \log\,Q_k + C_n\,\varepsilon_k^{-3}\log\,(\varepsilon_k^{-1}), \tag{5.2}$$

where [ ] stands for the integer part, and $M!!$ denotes the least common multiple of the natural numbers $1 \leq m \leq M$. Here, $C_n > 0$ is a large constant to be chosen later. Note that, by the prime number theorem, it is easy to see that $\log\,(Q_k) \lesssim \varepsilon_k^{-6/\,n-4} \leq \varepsilon_k^{-3}$ if $n \geq 6$, hence in that case the second term of (5.2) may be omitted.

We prove by induction that $(\varepsilon_k, Q_k, J_k)$ is regular for all k in the sense described in the introduction. So assume on the contrary that $(\varepsilon_{k-1}, Q_{k-1}, J_{k-1})$ is regular, but $(\varepsilon_k, Q_k, J_k)$ is not. Then there exists a sequence: $\lambda_1, \lambda_2, \ldots, \lambda_{J_k}$ with $10\lambda_j < \lambda_{j+1}$ a cube B_R of size $R \geq 10\,\lambda_{J_k}^{1/2}$, and a set $A \subset B_R$ with $|A| \geq \varepsilon_k R^n$, such that for all $1 \leq j \leq J_k$

$$\lambda_j\,Q_k^2 \notin d^2(A). \tag{5.3}$$

First we show that for every $L \geq (\lambda_{J_k/2})^{1/2}\,Q_k$ and for every cube $B_L \subset B_R$ of size L and $s \in \mathbb{Z}^n$, one has

$$|A'| = |\{m \in \mathbb{Z}^n : q_k m + s \in A \cap B_L\}| \geq \frac{10\varepsilon}{9}\,\frac{L^n}{q_k^n}. \tag{5.4}$$

Indeed, otherwise the set A' is contained in a cube $B_{R'}$ of size $R' = L/q_k$ of density at least ε_{k-1}. Since $R' = L/q_k > \lambda_{J_k/2}^{1/2} \geq 10\lambda_{J_{k-1}}^{1/2}$, by induction, there is a $j \leq J_{k-1}$ such that: $\lambda_j Q_{k-1}^2 \in d^2(A')$. Then $\lambda_j Q_{k-1}^2 q_k^2 = \lambda_j Q_k^2 \in d^2(A)$ contradicting our indirect assumption.

Let $J_k/2 \leq j \leq J_k$. To get in agreement with the notations of the previous sections, let $\varepsilon = \varepsilon_k$, $q = q_k$, $Q = Q_k$. If the constant C_n is chosen large enough then, by (5.1) and (5.2), the conditions of inequality (3.12) are satisfied for $\lambda = \lambda_j Q^2$ and $L' = L'_j = \delta\lambda^{1/2}$ with $\delta = \varepsilon^{6(n-1)+3/2}$. Thus

$$N_1(A, \lambda, Q, L') \leq \frac{c_n}{2}\,\varepsilon^3 R^n \lambda^{n/2-1}. \tag{5.5}$$

Let $\lambda^{1/2} \leq L_j \leq 2\lambda^{1/2}$ be chosen such that R/L_j is an integer. Then inequality (4.17) applies with $L = L_j$, thus

$$N_2(A, \lambda, Q, L) \geq c_n\,\varepsilon^3 R^n \lambda^{n/2-1}. \tag{5.6}$$

By our indirect assumption, $N(A, \lambda) = 0$ for each $\lambda = \lambda_j Q^2$, $J_k/2 \leq j \leq J_k$. We decompose the integral $N(A, \lambda)$ into three terms, as described in the introduction

$$N(A, \lambda) = N_1(A, \lambda, Q, L') + N_2(A, \lambda, Q, L) + N_3(A, \lambda, Q, L, L'), \tag{5.7}$$

where $N_3(A, \lambda, Q, L, L')$ is defined by the above equation. Thus by (5.5) and (5.6) one has

$$|N_3(A, \lambda_j Q^2, L_j, L'_j)| = \left| \int_{\Pi^n} |\hat{1}_A(\xi)|^2 \hat{\sigma}_\lambda(\xi) \Phi_j(\xi)\, d\xi \right| \geq \frac{c_n}{2}\,\varepsilon^3 R^n \lambda^{n/2-1}, \tag{5.8}$$

where

$$\Phi_j(\xi) = \sum_{l \in \mathbb{Z}^n} \hat{\psi}(L'_j(\xi - l/Q)) - \hat{\psi}(L_j(\xi - l/Q)). \tag{5.9}$$

Note that the integral $N_3(A, \lambda_j Q^2, L_j, L'_j)$ captures the contribution of the region: $\{\xi :\ L_j^{-1} \leq |\xi - l/Q| \leq \delta^{-1} L_j^{-1},\ l \in \mathbb{Z}^n\}$ to the integral $N(A, \lambda)$. Since $|\hat{\sigma}_\lambda(\xi)| \lesssim \lambda^{n/2\,-1}$ one has

$$N_3(j) := \int_{\Pi^n} |\hat{1}_A(\xi)|^2 |\Phi_j(\xi)| \geq c_n\,\varepsilon^3 R^n. \tag{5.10}$$

On the other hand, the integrands are essentially supported on disjoint sets, in fact one has

$$\sum_{J_k/2 \leq j \leq J_k} |\Phi_j(\xi)| \lesssim \log(\varepsilon^{-1}) \tag{5.11}$$

which implies

$$\sum_{J_k/2\leq j\leq J_k} N_3(j) \lesssim \log(\varepsilon^{-1})R^n. \tag{5.12}$$

This clearly contradicts (5.10) as J_k has been chosen to satisfy:

$$J_k > C_n\, \varepsilon_k^{-3} \log(\varepsilon_k^{-1})$$

with a large enough constant C_n. Finally, to see (5.11) first note that the functions:

$$\sum_{J_k/2\leq j\leq J_k} \hat{\psi}(L'_j(\xi - l/Q)) - \hat{\psi}(L_j(\xi - l/Q))$$

have disjoint supports for different values of l. Thus it is enough to show that for fixed l, for $\eta = \xi - l/Q$:

$$\sum_{J_k/2\leq j\leq J_k} |\hat{\psi}(L'_j\eta) - \hat{\psi}(L_j\eta)| \lesssim \sum_{j:\ L'_j|\eta|<1} \min(L_j|\eta|, 1)$$
$$\lesssim \sum_{j:\ L_j|\eta|<1} L_j|\eta| \ + \sum_{j:\ 1\leq L_j|\eta|<\delta^{-1}} 1 \lesssim \log(\varepsilon^{-1}).$$

This follows using the properties of $\hat{\psi}$ given in (3.10) and the fact that $L'_j \leq \delta L_j$. We have reached a contradiction to our indirect assumption and Theorem 2 is proved. ∎

It is not hard to estimate the size of Q_ε for any given $\varepsilon > 0$. Indeed if $\varepsilon_k \leq \varepsilon < \varepsilon_{k-1}$ then we take $Q_\varepsilon = Q_k$. Now $Q_k = \prod_{l=1}^k q_l$ where $q_l = M_l\,!! \leq \exp(C_n\,\varepsilon_l^{-6/\,n-4})$. It follows $Q_\varepsilon \leq \exp(C_n\,\varepsilon^{-6/\,n-4})$ for some constant C_n depending only on the dimension n.

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